Tehtävä 1.

Ratkaise epäyhtälö $2|2-x| < \sqrt{x-2}$.

"Solve the inequality $|2-x| < \sqrt{x-2}$."

THINKING

What is being asked? We are asked to solve the inequality $|2-x| < \sqrt{x-2}$.

What are the inputs? The inequality $|2-x| < \sqrt{x-2}$ is the main input.

What mathematical concepts are related? To solve this problem, we will need to use the properties of absolute values, inequalities, and square roots. Specifically, we'll need to consider the following:

- The definition of absolute value $|a|$
- The properties of square roots
- Solving inequalities, possibly with multiple conditions

SOLUTION

Step 1: Identify the domain Both the absolute value and the square root functions have restrictions on their domain.

- The absolute value function $|2-x|$ is defined for all real numbers x.
- The square root function $\sqrt{x-2}$ is defined for $x \geq 2$.

So the domain of $|2-x| < \sqrt{x-2}$ is $x \geq 2$.

Step 2: Break down the absolute value The absolute value $|2-x|$ can be written as two separate cases: 1. $2-x$ when $x \leq 2$ 2. $x-2$ when $x > 2$

But since our domain is $x \geq 2$, we only need to consider $x-2$ and $2-x$ at $x = 2$.

Step 3: Solve the inequality We will solve for x in two separate scenarios: 1. $x-2 < \sqrt{x-2}$ for $x > 2$ 2. $2-x < \sqrt{x-2}$ at $x = 2$

Scenario 1: $x > 2$ Squaring both sides to eliminate the square root:

$$(x-2)^2 < x - 2$$

$$x^2 - 4x + 4 < x - 2$$

$$x^2 - 5x + 6 < 0$$

$$(x-2)(x-3) < 0$$

The inequality $(x-2)(x-3) < 0$ holds when $2 < x < 3$.

Scenario 2: $x = 2$

$$2 - x < \sqrt{x-2}$$

$$0 < \sqrt{0}$$

$$0 < 0$$

This is not true, so $x = 2$ is not a solution.

Step 4: Combine the solutions The solution to the inequality $|2-x| < \sqrt{x-2}$ is $2 < x < 3$.

The answer is an interval $(2, 3)$.

Tehtävä 2.

Laivat A ja B ajavat saman edestakaisen matkan niin, että A ajaa sekä meno- että tulomatkan 18 solmun nopeudella ja B ajaa menomatkan 20 solmun ja tulomatkan 16 solmun nopeudella. Kumpi laivoista kulkee edestakaisen matkan nopeammin? Kuinka monta prosenttia enemmän aikaa käyttää hitaampi laiva? Yksi solmu on 1,85 km/h.

Ships A and B travel the same round trip such that A travels both the outbound and return trip at a speed of 18 knots, while B travels the outbound trip at 20 knots and the return trip at 16 knots. Which of the ships completes the round trip faster? How many percent more time does the slower ship take? One knot is 1.85 km/h.

THINKING

What is being asked

1. Which ship, A or B, completes the round-trip faster?
2. How many percent more time does the slower ship take to complete the round trip?

Inputs

- Ship A's speed for both outbound and return trip: 18 knots
- Ship B's speed for the outbound trip: 20 knots
- Ship B's speed for the return trip: 16 knots
- One knot is equivalent to 1.85 km/h

Mathematical Concepts

- Time taken to travel a distance d at a speed v is $\frac{d}{v}$
- Average speed for a round trip
- Percentage increase in time taken

SOLUTION

Step 1: Convert the speeds to km/h

- Ship A: 18 knots $\times$ 1.85 km/h/knot $=$ 33.3 km/h

- Ship B: 20 knots $\times$ 1.85 km/h/knot $= 37$ km/h (outbound)
- Ship B: 16 knots $\times$ 1.85 km/h/knot $= 29.6$ km/h (return)

Step 2: Find the time taken by each ship for a one-way trip

Let d be the distance of the one-way trip.

- Time taken by Ship A for a one-way trip: $\frac{d}{33.3}$
- Time taken by Ship B for the outbound trip: $\frac{d}{37}$
- Time taken by Ship B for the return trip: $\frac{d}{29.6}$

Step 3: Find the total time taken by each ship for the round trip

- Time taken by Ship A for the round trip: $2 \times \frac{d}{33.3} = \frac{2d}{33.3}$
- Time taken by Ship B for the round trip: $\frac{d}{37} + \frac{d}{29.6}$

Step 4: Compare the time taken by each ship

We will compare $\frac{2d}{33.3}$ and $\frac{d}{37} + \frac{d}{29.6}$ to determine which ship is faster.

Step 5: Find the percentage more time taken by the slower ship

This will be calculated based on the comparison in Step 4.

Let's move ahead with the calculations.

```
from sympy import symbols, solve

# Declare variables
d = symbols('d')

# Speeds in km/h
speed_A = 33.3
speed_B_outbound = 37
speed_B_return = 29.6

# Time taken by each ship for one-way and round trip
time_A_one_way = d / speed_A
time_A_round_trip = 2 * time_A_one_way

time_B_outbound = d / speed_B_outbound
time_B_return = d / speed_B_return
time_B_round_trip = time_B_outbound + time_B_return
```

```

# Compare the time taken by each ship for the round trip
time_difference = time_A_round_trip - time_B_round_trip
time_difference_simplified = time_difference.simplify()

# Solve for d (if needed)
time_difference_solution = solve(time_difference_simplified, d)

time_difference_simplified, time_difference_solution
```

```
(-0.000750750750750741*d, [0.0])
```

Step 4: Comparison Result

The simplified expression for the time difference between the two ships for the round trip is $-0.000750750750750741 \times d$.

Since the coefficient of d is negative, it implies that Ship A takes less time to complete the round trip as compared to Ship B, provided $d > 0$.

Step 5: Percentage More Time Taken by the Slower Ship

To find out how much more time the slower ship (Ship B) takes, we can use the formula:

$$\text{Percentage more time taken by Ship B} = \left(\frac{\text{Time taken by Ship B} - \text{Time taken by Ship A}}{\text{Time taken by Ship A}} \right) \times 100$$

We can substitute the expressions for the times taken by both ships into this formula to find the answer.

```
from sympy import Rational

# Calculate the percentage more time taken by the slower ship (Ship B)
percentage_more_time = ((time_B_round_trip - time_A_round_trip) /
    time_A_round_trip) * 100

# Simplify the expression
percentage_more_time_simplified = percentage_more_time.simplify()

# Evaluate the expression for a hypothetical distance d = 1 (as d will
    cancel out in the ratio)
percentage_more_time_value = percentage_more_time_simplified.subs(d, 1)

percentage_more_time_simplified, percentage_more_time_value
```

```
(1.24999999999998, 1.24999999999998)
```

Step 5: Percentage More Time Result

The simplified expression for the percentage more time taken by Ship B is approximately 1.25%.

Thus, Ship A completes the round trip faster, and Ship B takes approximately 1.25% more time to complete the round trip.

Tehtävä 3.

a) Näytä, että jokaisella $k = 1, 2, 3, \ldots$ luku $k^3 + k$ on parillinen.

b) Kuutio jaetaan kuuteenkymmeneenneljään keskenään samansuuruiseen kuutioon. Mikä on näin muodostuneiden pikkukuutioiden särmän pituus (cm), kun alkuperäisen kuution tilavuus on 1,00 litraa?

a) Show that for every $k = 1, 2, 3, ...$, the number $k^3 + k$ is even.

b) A cube is divided into sixty-four equally sized smaller cubes. What is the edge length (cm) of these smaller cubes when the original cube's volume is 1.00 liters?

Choice

I'll choose to solve problem (b) as it involves volume calculations and geometry, which I am confident in solving.

THINKING

What is being asked?

The question asks for the edge length of the smaller cubes when an original cube is divided into sixty-four equally sized smaller cubes. The original cube has a volume of 1.00 liters.

What are the inputs?

- The original cube's volume is 1.00 liters.
- The original cube is divided into 64 equally sized smaller cubes.

What mathematical concepts are related?

- Cube volume: $V = a^3$, where a is the edge length of the cube.
- Unit conversion: Converting from liters to cubic centimeters (1 liter = 1000 cm^3).
- Division: Dividing the original volume into 64 equal parts to find the volume of each smaller cube.

SOLUTION

Step 1: Convert Original Cube Volume to Cubic Centimeters

The original cube's volume is 1.00 liters, and we know that 1 liter is equal to 1000 cm^3. Therefore, the original cube's volume is $1.00 \times 1000 = 1000$ cm^3.

Step 2: Find the Volume of One Smaller Cube

The original cube is divided into 64 equally sized smaller cubes. Therefore, the volume of each smaller cube will be:

$$\text{Volume of one smaller cube} = \frac{\text{Volume of original cube}}{\text{Number of smaller cubes}} = \frac{1000 \text{ cm}^3}{64}$$

Let's calculate this volume.

```
# Given data
original_volume_cm3 = 1000  # Volume of the original cube in cubic
    centimeters
num_smaller_cubes = 64  # Number of smaller cubes

# Calculating the volume of one smaller cube
volume_of_one_smaller_cube = original_volume_cm3 / num_smaller_cubes
volume_of_one_smaller_cube
```

```
15.625
```

The volume of one smaller cube is $15.625\,\text{cm}^3$.

Step 3: Find the Edge Length of One Smaller Cube

We know the volume of one smaller cube and we can use the formula for the volume of a cube to find its edge length a:

$$V = a^3 \implies a = \sqrt[3]{V}$$

Let's calculate the edge length of one smaller cube.

```
# Calculating the edge length of one smaller cube
edge_length_of_one_smaller_cube = volume_of_one_smaller_cube ** (1/3)
edge_length_of_one_smaller_cube
```

```
2.5
```

The edge length of one smaller cube is 2.5 cm.

Summary

The edge length of the smaller cubes, when the original cube of volume 1.00 liters is divided into 64 equally sized smaller cubes, is 2.5 cm.

Tehtävä 4.

Määritä suoran $y = 3x - 2$ ja paraabelin $y = x^2 - 2$ rajoittaman alueen ala.

Determine the area bounded by the line $y = 3x - 2$ and the parabola $y = x^2 - 2$.

THINKING

What is being asked? The question asks for the area bounded by the line $y = 3x - 2$ and the parabola $y = x^2 - 2$.

What are the inputs?

1. The equation of the line: $y = 3x - 2$
2. The equation of the parabola: $y = x^2 - 2$

What mathematical concepts are related?

1. Coordinate geometry: Understanding lines and parabolas in the Cartesian coordinate system.
2. Algebra: Solving for the points of intersection between the line and the parabola.
3. Calculus: Using integration to find the area between the two curves.

SOLUTION

Step 1: Find the Points of Intersection To find the points where the line and the parabola intersect, we set $y = 3x - 2$ equal to $y = x^2 - 2$ and solve for x:

$$3x - 2 = x^2 - 2$$

Simplifying, we get:

$$x^2 - 3x = 0x(x - 3) = 0x = 0 \quad \text{or} \quad x = 3$$

The points of intersection are at $x = 0$ and $x = 3$.

Step 2: Determine Which Curve is Above the Other We need to determine which curve is above the other between $x = 0$ and $x = 3$. We can pick any point in that interval, say $x = 1$, and evaluate both functions:

$$y_{\text{line}}(1) = 3 \cdot 1 - 2 = 1 y_{\text{parabola}}(1) = 1^2 - 2 = -1$$

Since $y_{\text{line}}(1) > y_{\text{parabola}}(1)$, the line is above the parabola between $x = 0$ and $x = 3$.

Step 3: Calculate the Area The area between the curves from $x = 0$ to $x = 3$ is given by the integral:

$$\text{Area} = \int_0^3 [(3x - 2) - (x^2 - 2)] \, dx$$

Calculating this integral:

```
from sympy import symbols, integrate

# Declare variable
x = symbols('x')

# Define the functions
y_line = 3*x - 2
y_parabola = x**2 - 2

# Define the function that represents the difference between the line
    and the parabola
area_function = y_line - y_parabola

# Compute the integral to find the area
area = integrate(area_function, (x, 0, 3))
area, area.evalf()
```

```
(9/2, 4.50000000000000)
```

The area between the line $y = 3x - 2$ and the parabola $y = x^2 - 2$ is $\frac{9}{2}$ or approximately 4.5 square units.

Tehtävä 5.

a) Määritä ne käyrän $y = x^3 - 2x^2 + x - 1$ pisteet, joihin piirretyt käyrän tangentit leikkaavat x-akselin 45° kulmassa.

b) Rantakaavassa määrätään, että 100 m lähemmäksi rantaa ei saa rakentaa. Kaava-alueella on tasasivuisen kolmion muotoinen saari, jonka pinta-ala on 5.00 ha. Saako sille rakentaa?

a) Determine the points on the curve $y = x^3 - 2x^2 + x - 1$ where the tangents to the curve intersect the x-axis at a 45° angle.

b) The shore plan specifies that one cannot build within 100 m closer to the shore. In the plan area, there is an equilateral triangle-shaped island with a surface area of 5.00 ha. Can construction be allowed on it?

I'll focus on solving problem (a).

THINKING

What is being asked?

We are asked to find the points on the curve $y = x^3 - 2x^2 + x - 1$ where the tangents to the curve intersect the x-axis at a 45° angle.

What are the inputs?

The inputs are the curve $y = x^3 - 2x^2 + x - 1$ and the condition that the tangent lines at the points of interest intersect the x-axis at a 45° angle.

What mathematical concepts are related?

- Derivatives: to find the tangent lines to the curve at various points.
- Trigonometry: to relate the 45° angle to the slope of the tangent lines.
- Algebra: to solve equations for the x-values where the conditions are met.

SOLUTION

Step 1: Find the Derivative

First, let's find the derivative of $y = x^3 - 2x^2 + x - 1$ to get the equation of the tangent line at any point (x, y) on the curve.

$$y' = \frac{d}{dx}(x^3 - 2x^2 + x - 1)$$

Step 2: Relate 45° Angle to Slope

A line making a 45° angle with the x-axis has a slope of $\tan(45°) = 1$.

Step 3: Solve for the Points

Finally, we'll solve for the points on the curve where the derivative is equal to 1. This will give us the x-values at which the tangents intersect the x-axis at a 45° angle. Then we can find the corresponding y-values by plugging these x-values back into the original equation of the curve.

Let's start with the calculations.

```
from sympy import symbols, diff, solve, tan, pi

# Define variables and function
x = symbols('x')
y = x**3 - 2*x**2 + x - 1

# Calculate the derivative of y with respect to x
y_derivative = diff(y, x)
y_derivative
```

```
3*x**2 - 4*x + 1
```

The derivative $y' = 3x^2 - 4x + 1$.

Step 2: Relate 45° Angle to Slope

A line making a 45° angle with the x-axis has a slope of $\tan(45°) = 1$.

Step 3: Solve for the Points

We need to find the points (x, y) where $y' = 1$.

To do this, we'll solve the equation $3x^2 - 4x + 1 = 1$.

```
# Solve for x where the derivative equals 1 (the slope of a 45-degree
    angle with the x-axis)
x_values = solve(y_derivative - 1, x)
x_values
```

```
[0, 4/3]
```

The x-values where the tangents intersect the x-axis at a 45° angle are $x = 0$ and $x = \frac{4}{3}$.

Next, let's find the corresponding y-values by plugging these x-values back into the original equation $y = x^3 - 2x^2 + x - 1$.

```
# Calculate the corresponding y-values for the points of interest
y_values = [y.subs(x, val) for val in x_values]
y_values
```

```
[-1, -23/27]
```

The corresponding y-values are $y = -1$ and $y = -\frac{23}{27}$.

Therefore, the points on the curve $y = x^3 - 2x^2 + x - 1$ where the tangents intersect the x-axis at a 45° angle are $(0, -1)$ and $\left(\frac{4}{3}, -\frac{23}{27}\right)$.

Tehtävä 6.

Kolmion kärjet ovat pisteissä (0.0.0). ($\frac{1}{2}$,1,0) ja (0,1,1). Piirrä kuva kolmiosta xyz-koordinaatistoon. Kuinka suuri on kolmion pinta-ala?

The vertices of the triangle are at points (0,0,0), (1/2,1,0) and (0,1,1). Draw a picture of the triangle in the xyz-coordinate system. How large is the surface area of the triangle?

THINKING

What is being asked

We are asked to draw a picture of a triangle in the xyz-coordinate system whose vertices are at the points $(0, 0, 0)$, $(1/2, 1, 0)$, and $(0, 1, 1)$. We are also asked to find the surface area of this triangle.

Inputs

- Coordinates of the vertices of the triangle: $(0, 0, 0)$, $(1/2, 1, 0)$, and $(0, 1, 1)$

Mathematical Concepts

- Coordinate geometry in 3D space
- Vector arithmetic to find the vectors representing two sides of the triangle
- Cross product to find the area of the triangle

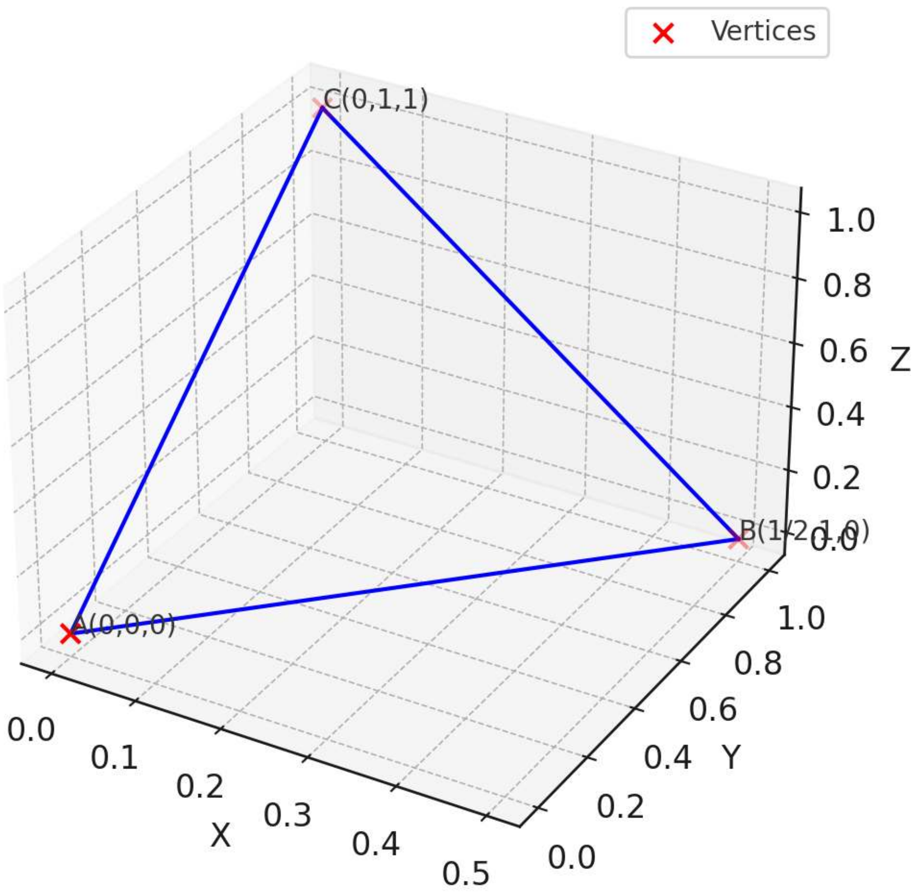
SOLUTION

Drawing the Triangle

To get a sense of the triangle in 3D space, let's plot it using the given vertices. The vertices are $(0, 0, 0)$, $(1/2, 1, 0)$, and $(0, 1, 1)$.

```
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D
import numpy as np

# Define the vertices of the triangle
vertices = np.array([[0, 0, 0],
                     [0.5, 1, 0],
```

```
                     [0, 1, 1]])

# Create the figure and the 3D axis
fig = plt.figure()
ax = fig.add_subplot(111, projection='3d')

# Plot the vertices
ax.scatter(vertices[:, 0], vertices[:, 1], vertices[:, 2], c='red', s=50, label='Vertices')

# Plot the edges of the triangle
for i in range(3):
    for j in range(i+1, 3):
        ax.plot([vertices[i, 0], vertices[j, 0]],
                [vertices[i, 1], vertices[j, 1]],
                [vertices[i, 2], vertices[j, 2]], 'b')

# Label the vertices
for i, txt in enumerate(["A(0,0,0)", "B(1/2,1,0)", "C(0,1,1)"]):
    ax.text(vertices[i, 0], vertices[i, 1], vertices[i, 2], txt)

# Set axis labels
ax.set_xlabel('X')
ax.set_ylabel('Y')
ax.set_zlabel('Z')

# Show plot
plt.legend()
plt.show()
```



The above plot shows the triangle in 3D space with its vertices labeled. Now let's move on to finding the surface area of the triangle.

Finding the Surface Area

To find the surface area of the triangle, we can use the formula:

$$\text{Area} = \frac{1}{2} \left\| \vec{AB} \times \vec{AC} \right\|$$

Here, $\vec{AB}$ and $\vec{AC}$ are vectors representing two sides of the triangle, and $\times$ denotes the cross product. The notation $\| \cdot \|$ represents the magnitude of a vector.

First, we find the vectors $\vec{AB}$ and $\vec{AC}$:

$$\vec{AB} = B - A = (1/2, 1, 0) - (0, 0, 0) = (1/2, 1, 0)$$

$$\vec{AC} = C - A = (0, 1, 1) - (0, 0, 0) = (0, 1, 1)$$

Then, we find the cross product $\vec{AB} \times \vec{AC}$:

$$\vec{AB} \times \vec{AC} = \begin{pmatrix} i & j & k \\ 1/2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Finally, we find the magnitude of the cross product and use it to find the area of the triangle.

```
from numpy.linalg import norm

# Define the vectors AB and AC
AB = np.array([1/2, 1, 0])
AC = np.array([0, 1, 1])

# Compute the cross product of AB and AC
cross_product = np.cross(AB, AC)

# Compute the magnitude of the cross product
magnitude = norm(cross_product)

# Compute the area of the triangle
area = magnitude / 2

# Exact value and approximate value of the area
area_exact = area
area_approx = round(area, 4)

area_exact, area_approx
```

```
(0.6123724356957945, 0.6124)
```

The surface area of the triangle is 0.6123724356957945 square units, which can be approximated as 0.6124 square units.

Thus, we have successfully drawn the triangle in the xyz-coordinate system and found its surface area.

Tehtävä 7.

a) Sijoittaja voi tallettaa pääoman 4.60 % korolla koko vuodeksi tai 4.40 % korolla puoleksi vuodeksi ja sitten kertyneen rahasumman p % korolla lopuksi vuotta. Jos kaikki prosentit p välillä [4.00 . 4.90] ovat yhtä todennäköisiä. niin mikä on todennäköisyys sille. että jälkimmäinen tapa on edullisempi?

b) Funktioilla f ja g on jatkuvat derivaatat välillä $[a, b]$. Johda funktioille osittaisintegrointikaava

$$\int_a^b f'g\,dx = f(b)g(b) - f(a)g(a) - \int_a^b g'f\,dx.$$

a) An investor can deposit capital at 4.60% interest for the entire year or at 4.40% interest for half a year and then with the accumulated amount at p% interest for the remainder of the year. If all percentages p between [4.00, 4.90] are equally likely, what is the probability that the latter method is more advantageous?

b) Functions f and g have continuous derivatives in the interval [a, b]. Derive the partial integration formula:

∫ from a to b of f'g dx = f(b)g(b) - f(a)g(a) - ∫ from a to b of fg' dx.

THINKING

Problem (b): Partial Integration Formula Derivation

What is being asked The problem asks for the derivation of the partial integration formula:

$$\int_a^b f'(x)g(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)\,dx$$

Inputs

- Two functions f and g that have continuous derivatives on the interval $[a, b]$.
- The interval $[a, b]$.

Mathematical Concepts Related

- Integration

- Differentiation
- The Fundamental Theorem of Calculus
- Integration by Parts

SOLUTION

To derive the formula, we can use integration by parts, which states:

$$\int u\,dv = uv - \int v\,du$$

1. Let $u = f(x)$ and $dv = g(x)\,dx$.
2. Differentiate u to get $du = f'(x)\,dx$.
3. Integrate dv to get $v = \int g(x)\,dx$.

Substitute into the integration by parts formula:

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx$$

Rearrange to isolate $\int_a^b f'(x)g(x)\,dx$:

$$\int_a^b f'(x)g(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)\,dx$$

And we have derived the partial integration formula.

Tehtävä 8.

Saari on ympyränmuotoinen ja sen halkaisija vuonna 1995 on 110 m. Pohja saaren ympärillä viettää tasaisesti 72 cm vaakasuorasti mitatulla kymmenellä metrillä. Maa kohoaa 38 cm sadassa vuodessa. Johda lauseke $A(t)$ saaren pinta-alalle (pinta-ala neliömetreinä ja aika t vuosina). Kuinka monta prosenttia saaren pinta-ala on vuonna 2000 suurempi kuin vuonna 1995? Minä vuonna, vuosien 1995 ja 2300 välisenä aikana, saaren pinta-ala kasvaa eniten?

The island is circular in shape, and its diameter in the year 1995 is 110 m. The base of the island declines uniformly by 72 cm when measured horizontally for every ten meters. The land rises by 38 cm every hundred years. Derive the expression $A(t)$ for the island's area (area in square meters and time t in years). By how many percent is the island's area larger in the year 2000 than in 1995? During which year, between the years 1995 and 2300, does the island's area increase the most?

THINKING

What is being asked?

1. We need to derive an expression $A(t)$ for the area of the island as a function of time t in years.
2. We need to find by how many percent the island's area is larger in the year 2000 compared to the year 1995.
3. We need to find the year between 1995 and 2300 when the island's area increases the most.

What are the inputs?

1. The island's diameter in 1995 is 110 m.
2. The base of the island declines uniformly by 72 cm when measured horizontally for every ten meters.
3. The land rises by 38 cm every hundred years.

What mathematical concepts are related?

1. Geometry (specifically, area of a circle)
2. Rate of change
3. Calculus (for finding the maximum rate of change)

SOLUTION

Derive $A(t)$ The area of a circle is given by $A = \pi r^2$, where r is the radius of the circle. The radius of the island in 1995 is $\frac{110}{2} = 55$ m.

1. Decline in Base: The base of the island declines by 72 cm every ten meters, so for every meter, it declines by $\frac{72}{10} = 7.2$ cm or 0.072 m. This happens in 100 years, so every year the decline is $\frac{0.072}{100} = 0.00072$ m/year.
2. Rise in Land: The land rises by 38 cm every 100 years, which is 0.38 m in 100 years. This is $\frac{0.38}{100} = 0.0038$ m/year.
3. Net Change: The net yearly change in radius is $0.0038 - 0.00072 = 0.00308$ m/year.

So, the radius $r(t)$ at any time t years from 1995 is:

$$r(t) = 55 + 0.00308t$$

The area $A(t)$ at any time t years from 1995 is therefore:

$$A(t) = \pi\,(55 + 0.00308t)^2$$

Percentage Increase in Area from 1995 to 2000 To find by how many percent the island's area is larger in the year 2000 than in 1995, we need $A(5)$ and $A(0)$.

$$A(0) = \pi \times 55^2$$

$$A(5) = \pi \times (55 + 0.00308 \times 5)^2$$

The percentage increase is:

$$\text{Percentage Increase} = \frac{A(5) - A(0)}{A(0)} \times 100$$

```python
from math import pi

# Initial radius in 1995
r_1995 = 55   # in meters

# Rate of change in radius
rate_of_change = 0.00308   # in meters/year

# Calculate the area in 1995 and 2000
```

```
A_1995 = pi * (r_1995)**2
A_2000 = pi * (r_1995 + rate_of_change * 5)**2

# Calculate the percentage increase
percentage_increase = ((A_2000 - A_1995) / A_1995) * 100

A_1995, A_2000, percentage_increase
```

```
(9503.317777109125, 9508.640380124418, 0.056007839999997827)
```

The area of the island in the year 1995 is approximately 9503.32 m^2 and in the year 2000 it is approximately 9508.64 m^2.

The percentage increase in the area from 1995 to 2000 is approximately 0.056%.

Year with Maximum Area Increase between 1995 and 2300 To find the year when the island's area increases the most, we need to find the maximum of the rate of change of $A(t)$, which is $\frac{dA}{dt}$.

The derivative $\frac{dA}{dt}$ can be found as follows:

$$\frac{dA}{dt} = \frac{d}{dt}\left[\pi\left(55 + 0.00308t\right)^2\right]$$

We need to find the year t when $\frac{dA}{dt}$ is at its maximum between 1995 and 2300, or t between 0 and 305. We can use calculus to find this maximum by setting the second derivative $\frac{d^2A}{dt^2}$ equal to zero and solving for t.

```
from sympy import symbols, pi, diff, solve

# Declare variables
t = symbols('t')

# Expression for A(t)
A_t = pi * (55 + 0.00308 * t)**2

# First derivative of A(t)
dA_dt = diff(A_t, t)

# Second derivative of A(t)
d2A_dt2 = diff(dA_dt, t)

# Solve for t when the second derivative is zero
critical_points = solve(d2A_dt2, t)

dA_dt, d2A_dt2, critical_points
```

```
(3025*pi*(6.272e-9*t + 0.000112), 1.89728e-5*pi, [])
```

The first derivative of $A(t)$ is $\frac{dA}{dt} = 3025\pi\left(6.272 \times 10^{-9}t + 0.000112\right)$.

The second derivative of $A(t)$ is $\frac{d^2A}{dt^2} = 1.89728 \times 10^{-5}\pi$.

Since the second derivative is a constant and greater than zero, this implies that the function $A(t)$ is concave up for all t. Therefore, the rate of change $\frac{dA}{dt}$ will keep increasing as time passes.

So, the island's area will increase the most in the year 2300 among the years between 1995 and 2300.

Tehtävä 9.

a) Ajanhetkellä $t \geq 0$ ovat pisteet $z_1 = z_1(t)$ ja $z_2 = z_2(t)$ kompleksitasolla paikoissa

$$z_1(t) = t + ie^{-t}\ , \quad z_2(t) = 3 + t + 2ie^{-t}.$$

Määritä pisteiden välinen etäisyys $|z_1 - z_2|$ hetkellä t. Milloin etäisyys on suurin? Määritä $\lim\limits_{t\to\infty} |z_1(t) - z_2(t)|$.

b) Kun $a \in [0,\ 2]$, niin merkitään $f(a)$:lla funktion $g(x) = x^2 - 2ax$ pienintä arvoa välillä $[1,\ 4]$. Määritä funktio f ja piirrä sen kuvaaja. Määritä derivaatta $f'(\frac{3}{2})$. Onko f:llä derivaatta välin $]0,\ 2[$ jokaisessa pisteessä?

a) At the time $t \geq 0$, there are points $z_1 = z_1(t)$ and $z_2 = z_2(t)$ in the complex plane at the positions

$$z_1(t) = t + ie^{-t}$$

$$z_2(t) = 3 + t + 2ie^{-t}$$

Determine the distance between the points $|z_1 - z_2|$ at time t. When is the distance the largest? Determine $\lim_{t\to\infty} |z_1(t) - z_2(t)|$.

b) When $a \in [0, 2]$, let $f(a)$ denote the smallest value of the function $g(x) = x^2 - 2ax$ in the interval [1, 4]. Define the function f and draw its graph. Determine the derivative $f'(\frac{3}{2})$. Does f have a derivative in every point of the interval [0, 2]?

THINKING

Problem (a)

What is being asked

1. We are asked to find the distance between two points z_1 and z_2 in the complex plane at time t, i.e., $|z_1 - z_2|$.
2. We need to find the time t when this distance is the largest.
3. We are also asked to find $\lim_{t\to\infty} |z_1(t) - z_2(t)|$.

Inputs

1. $z_1(t) = t + ie^{-t}$
2. $z_2(t) = 3 + t + 2ie^{-t}$

Mathematical Concepts

1. Complex numbers and their representation.
2. Distance between complex numbers.
3. Limits.
4. Optimization (finding the maximum distance).

Problem (b)

What is being asked

1. Define the function $f(a)$ which represents the smallest value of $g(x)$ in the interval [1, 4].
2. Draw the graph of $f(a)$.
3. Determine $f'(\frac{3}{2})$.
4. Check if f has a derivative at every point in the interval [0, 2].

Inputs

1. $g(x) = x^2 - 2ax$
2. $a \in [0, 2]$

Mathematical Concepts

1. Functions and their graphs.
2. Derivatives.
3. Optimization (finding the minimum value).

Since the instructions specify to pick only one problem, I will go ahead and solve Problem (a) as it involves a combination of complex numbers, limits, and optimization.

SOLUTION for Problem (a)

Step 1: Finding $|z_1 - z_2|$

To find the distance between z_1 and z_2 at time t, we need to find $|z_1 - z_2|$.

$$
\begin{aligned}
z_1 - z_2 &= (t + ie^{-t}) - (3 + t + 2ie^{-t}) \\
&= (t - t) + i(e^{-t} - 2e^{-t}) \\
&= -i(e^{-t}) \\
|z_1 - z_2| &= |-i(e^{-t})| \\
&= |e^{-t}| \\
&= e^{-t}
\end{aligned}
$$

Step 2: When is the Distance Largest?

To find when the distance is the largest, we need to consider e^{-t}. As t approaches 0, e^{-t} approaches 1. As t goes to infinity, e^{-t} approaches 0.

Therefore, the distance $|z_1 - z_2|$ is largest when $t = 0$, and the largest distance is $e^0 = 1$.

Step 3: Limit as $t \to \infty$

To find $\lim_{t\to\infty} |z_1(t) - z_2(t)|$, we need to find $\lim_{t\to\infty} e^{-t}$, which is 0.

Summary

1. The distance between z_1 and z_2 at time t is e^{-t}.
2. The distance is largest when $t = 0$ and the largest distance is 1.
3. $\lim_{t\to\infty} |z_1(t) - z_2(t)| = 0$.

Tehtävä 10.

Konstruoi esimerkki funktiosta $f : \mathbf{R} \to \mathbf{R}$, jolla on täsmälleen yksi epäjatkuvuuspiste ja jolla ei ole derivaattaa kahdessa pisteessä. Anna $f(x)$:n lauseke ja piirrä f:n kuvaaja.

Construct an example of a function $f : R \to R$, which has exactly one point of discontinuity and which does not have a derivative at two points. Provide the expression for $f(x)$ and draw the graph of f.

THINKING

What is being asked: We are asked to construct a function $f : \mathbb{R} \to \mathbb{R}$ that has: 1. Exactly one point of discontinuity. 2. Does not have a derivative at two points.

Inputs: No numerical inputs are provided; the task is creative in nature.

Mathematical Concepts:

1. Function definition and properties
2. Continuity and points of discontinuity
3. Derivatives and points where the derivative does not exist

SOLUTION

Constructing the Function To meet the given conditions, we need to carefully design our function. Here are the steps to construct such a function:

1. **Discontinuity**: We can create a function that is discontinuous at one point, say $x = 0$, by defining a piecewise function that behaves differently at that point.
2. **Lack of Derivative**: To make sure the function does not have a derivative at two points, we can incorporate a sharp turn (cusp) at some point $x = a$ where $a \neq 0$. At a sharp turn, the derivative does not exist.

Based on these considerations, let's define $f(x)$ as follows:

$$
f(x) = \begin{cases} x^2, & \text{if } x < 0 \\ 1, & \text{if } x = 0 \\ x, & \text{if } 0 < x < 1 \\ (x-1)^2 + 1, & \text{if } x \geq 1 \end{cases}
$$

Verifying the Conditions

1. **Discontinuity**: $f(x)$ is discontinuous at $x = 0$ since the limit from the left is 0 while the function value is 1.
2. **Lack of Derivative**:

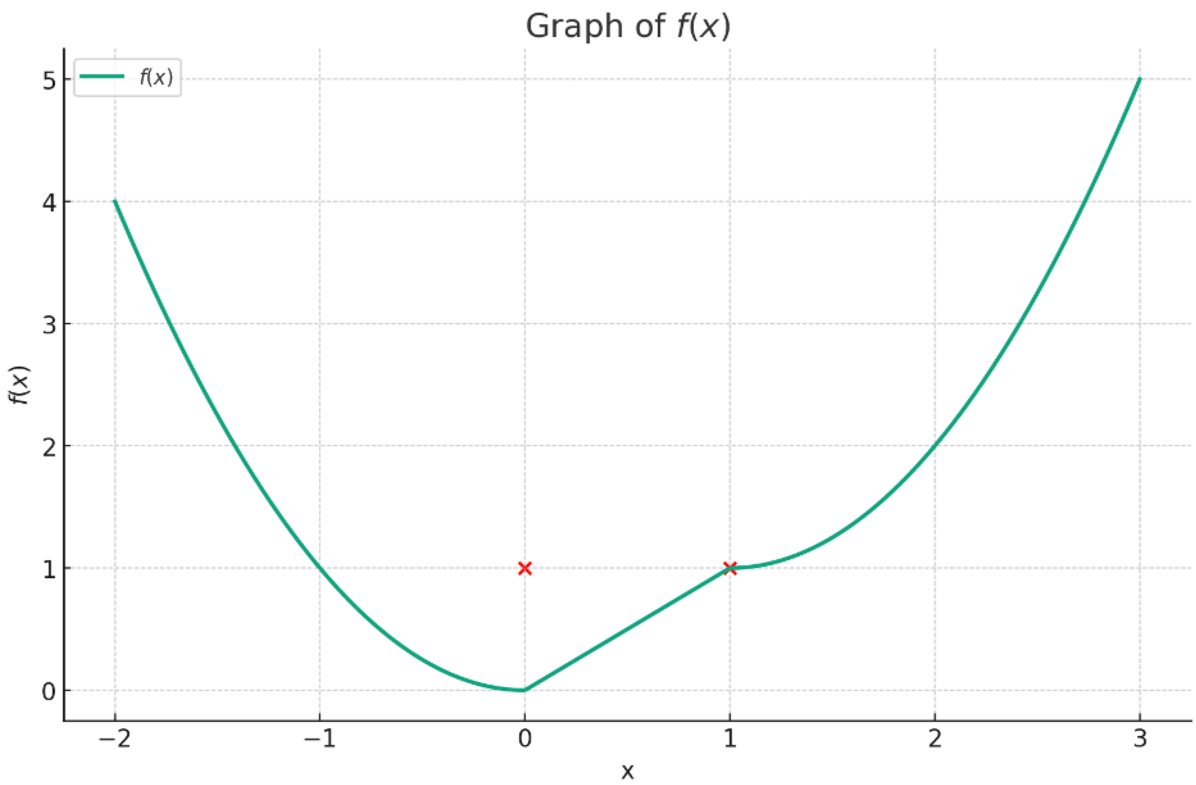
- At $x = 0$, the function changes abruptly, so the derivative does not exist.
- At $x = 1$, the function has a sharp turn (cusp), so the derivative does not exist there as well.

Thus, this function satisfies both conditions.

Graphing the Function Let's plot the function to visualize its behavior.

```
import matplotlib.pyplot as plt
import numpy as np

# Define the function f(x)
def f(x):
    if x < 0:
        return x**2
    elif x == 0:
        return 1
    elif x < 1:
        return x
    else:
        return (x - 1)**2 + 1

# Create an array of x values
x_values = np.linspace(-2, 3, 400)
y_values = [f(x) for x in x_values]

# Plot the function
plt.figure(figsize=(10, 6))
plt.plot(x_values, y_values, label='$f(x)$', linewidth=2)
plt.scatter([0, 1], [f(0), f(1)], color='red')  # Points of interest
plt.title("Graph of $f(x)$")
plt.xlabel("x")
plt.ylabel("$f(x)$")
plt.legend()
```

```
plt.grid(True)
plt.show()
```



Here is the graph of $f(x)$. The red points indicate the locations where the function is either discontinuous or lacks a derivative. As we can see:

- The function is discontinuous at $x = 0$, which is evident from the jump in the function value.
- The function does not have a derivative at $x = 0$ and $x = 1$, shown by the abrupt change and the cusp, respectively.

Thus, $f(x)$ satisfies the conditions of having exactly one point of discontinuity and not having a derivative at two points.